Harmonic Oscillator Coherent States

1D quantum harmonic oscillator

$$\hat{\mathcal{H}} = \frac{\hat{p}^2}{2m} + \frac{m\,\omega^2}{2}\,\hat{x}^2$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + \frac{1}{2}m\omega^2 x^2\psi(x) = E\psi(x)$$

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega, \text{ where } n = 0, 1, 2, 3, \ldots,$$

$$\psi_n(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^n n!}}\, H_n\left(\sqrt{\frac{m\omega}{\hbar}}\,x\right) e^{-m\omega^2 x^2/2\hbar}$$

$$\hat{a}_+ = \frac{1}{\sqrt{2m\hbar\omega}}(-i\hat{p} + m\omega\hat{x}); \quad \hat{a}_- = \frac{1}{\sqrt{2m\hbar\omega}}(+i\hat{p} + m\omega\hat{x})$$

Adopting the notation of Griffiths QM, 3rd edition

$\hat{a}_+$ and $\hat{a}_-$ are non-Hermitian operators

$$\hat{a}_+\psi_n(x) = \sqrt{n+1}\,\psi_{n+1}(x)$$

$$\hat{a}_-\psi_n(x) = \sqrt{n}\,\psi_{n-1}(x)$$

$$\hat{N}\psi_n = \hat{a}_+\hat{a}_-\psi_n = n\psi_n$$

$$\hat{\mathcal{H}} = \hbar\omega\left(\hat{a}_+\hat{a}_- + \frac{1}{2}\right)$$

$$[\hat{\boldsymbol{x}}, \hat{\boldsymbol{p}}] = \boldsymbol{i}\hbar$$

$$[\hat{a}_-, \hat{a}_+] = \mathbf{1}$$

$$\psi_n(x) = \frac{1}{\sqrt{n!}}(\hat{a}_+)^n\psi_0(x)$$

Coherent State of the Quantum Harmonic Oscillator

$$|\alpha\rangle = C\left(\psi_0(x) + \frac{\alpha}{\sqrt{1!}}\psi_1(x) + \frac{\alpha^2}{\sqrt{2!}}\psi_2(x) + \frac{\alpha^3}{\sqrt{3!}}\psi_3(x) + \cdots\right)$$

α is an arbitrary complex number

$C = e^{-|\alpha|^2/2}$ by normalization

The state can also be written as: $|\alpha\rangle = e^{-|\alpha|^2/2}\, e^{\alpha \hat{a}_+}|0\rangle$

This state is an eigenfunction of the annihilation operator $\hat{a}_-|\alpha\rangle = \alpha|\alpha\rangle$

$$\langle\alpha|\hat{a}_-|\alpha\rangle = \alpha$$

$\langle\alpha|\hat{a}_+\hat{a}_-|\alpha\rangle = |\alpha|^2 = \langle n\rangle$ the mean number of excitations in the coherent state

The uncertainty in the number of particles: $\Delta n = \sqrt{\langle n^2\rangle - \langle n\rangle^2} = |\alpha|$

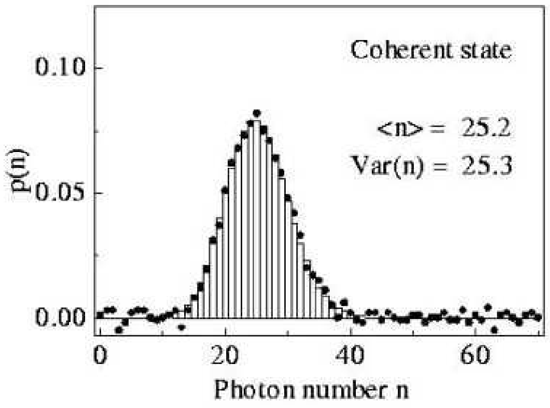
Coherent states do not have a fixed number of particles. However $\frac{\Delta n}{n} = \frac{1}{\sqrt{n}} \rightarrow 0$ in the thermodynamic limit

Coherent State of the Quantum Harmonic Oscillator

Statistical distribution of the occupation number $\langle n \rangle = |\alpha|^2$

$$P(n) = |\langle n | \alpha \rangle|^2 = \frac{|\alpha|^{2n} e^{-|\alpha|^2}}{n!}$$

Poisson distribution



The probability of detecting n photons, the photon number distribution, of a coherent state. As is necessary for a Poissonian distribution the mean photon number is equal to the variance of the photon number distribution. Bars refer to theory, dots to experimental values.

https://en.wikipedia.org/wiki/Coherent_state

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{(\alpha a^\dagger)^n}{n!} |0\rangle$$

Coherent State of the Quantum Harmonic Oscillator

If we write: $\alpha = |\alpha| e^{i\theta}$ then,

$$|\alpha\rangle = e^{-|\alpha|^2/2}\left(\psi_0(x) + e^{i\theta}\frac{|\alpha|}{\sqrt{1!}}\psi_1(x) + e^{i2\theta}\frac{|\alpha|^2}{\sqrt{2!}}\psi_2(x) + \cdots\right)$$

Note that the same phase θ appears in each term (coherent state),
as opposed to a randomly fluctuating phase in each term (incoherent state)

One can show that the θ-derivative is equivalent to the number operator: $\frac{1}{i}\frac{\partial}{\partial\theta}|\alpha\rangle = \hat{n}|\alpha\rangle$

Hence we can define: $\hat{n} = \frac{1}{i}\frac{\partial}{\partial\theta}$, hence the number and phase of the wavefunction are conjugate variables

There is a resulting uncertainty relation: $\Delta n\, \Delta\theta \geq \frac{1}{2}$

The coherent state $|\alpha\rangle$ is a superposition of all possible occupation numbers, with Δn large, hence it must have $\Delta\theta \to 0$